Department of Mathematics
University of Toronto
MAT 315H1S
Term-Test
Duration: 105 minutes

Examiner: Brooke Feigon Date: February 10, 2009

LAST NAME:_____

FIRST NAME:_____

STUDENT NUMBER:_____

SIGNATURE: _____

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

NOTES

- There are five questions, each of which is worth 20 marks.
- Before you start, check that this test has 11 pages, including this cover page.
- Explain and justify your work. If you need more space use the back of the page.
- **NO calculators, scrap paper, or other aids permitted.**
- **DO NOT tear any pages from this test.**

FOR MARKER ONLY	
Question	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

1. (a) What are the last two digits in the ordinary decimal representation of $3^{402} + 7^{81}$?

1. **continued**

   (b) Find all solutions to $x^3 + 4x + 8 \equiv 0 \pmod{15}$.

2. (a) Assume that x, y and z are natural numbers such that $x^2 + y^2 = z^2$. Prove that at least one of x, y or z must be divisible by 5.

2. **continued**

(b) Let a, b and c be fixed integers. Prove that a divides bc if and only if a divides $\gcd(a, b)\gcd(a, c)$.

3. (a) Find all solutions of the congruence $57x \equiv 6 \pmod{105}$.

3. **continued**

(b) Let $\gcd(a, b) = 1$. Prove that for any positive integer k the arithmetic progression

$$a + b, a + 2b, a + 3b, \ldots$$

contains k consecutive terms that are composite.

4. (a) Prove that any positive integer of the form $4k^2 + 1$ has a prime factor of the form $4n + 1$.

4. **continued**

(b) Prove that there are infinitely many primes of the form $4n + 1$.

5. (a) Determine whether there are solutions to the following system of congruences. If there are solutions, find the smallest positive integer that satisfies the system of congruences.

$$x \equiv 4 \pmod 6$$
$$x \equiv 5 \pmod{10}$$
$$x \equiv 3 \pmod 9$$

5. **continued**

(b) Determine whether there are solutions to the following system of congruences. If there are solutions, find the smallest positive integer that satisfies the system of congruences.

$$x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 5$$
$$x \equiv 4 \pmod 6$$
$$x \equiv 3 \pmod 7$$